MODERN EUCLID BOOK VI
SIMILARITY

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Abstract. This installment sets out the basis for numerical area, and the area of a triangle. The algebraic formula for area may be used to shorten some of Euclid's arguments. This document creates a modern synopsis of his first four propositions from Book VI, which is as far as we need to go to arrive at the main result on similar triangles.

1. Introduction

Modern Definition 1. Two triangles are *similar* if there is a correspondence between their vertices such that corresponding angles are congruent.

The notation $\triangle ABC \sim \triangle DEF$ means that $\triangle ABC$ is similar to $\triangle DEF$, with the vertices corresponding as indicated, so that:

- $\angle ABC \cong \angle DEF$,
- $\angle ACB \cong \angle DFE$, and
- $\angle BAC \cong \angle EDF$.

Notice that Euclid's Definition I.20 says that an *equilateral triangle* is one in which all three sides are of the same length, and one would think that an equiangular triangle would be one in which all the angles are equal, which would in turn imply that the triangle is equilateral. However, in Book VI, Euclid uses the phrase *equiangular triangles* to indicate a pair of triangles in which corresponding angles equal each other; that is, ones which we would call similar. Euclid's Definition VI.1 says that similar triangles "have their angles severally equal and the sides about equal angles proportional". Our goal is to prove that equal angles imply proportional sides.

Modern Proposition 1. Angle-Angle Similarity (AA)
If two triangles have a correspondence between their vertices such that two pairs of corresponding angles are congruent, then so is the third pair, and the triangles are similar.

Proof. The measure of the third angle is $180°$ minus the measure of the sum of the measures of the other two angles. Since this sum is the same in the case of the two pairs of corresponding angles, the angles in the third pair of corresponding angles must also be equal. □

We wish to prove the Modern Similarity Theorem, which says that given a pair of similar triangles, the ratios between the corresponding sides are equal. We will follow Euclid's path to do so, but will rephrase some of his proofs to use algebra. First, we review some of Euclid previous theorems.

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2. Area from Book I

In a plane, there exists a real number which is the distance between any two points. Let $d(A, B)$ denote the distance from point A to point B. Distance satisfies these properties.

(D1) Positive Definiteness: $d(A, B) \geq 0$, and $d(A, B) = 0$ if and only if $A = B$
(D2) Symmetry: $d(A, B) = d(B, A)$
(D3) Triangle Inequality: $d(A, B) + d(B, C) \geq d(A, C)$

The distance from a point to a line is the shortest distance. This is equal to the length of the segment which is perpendicular to the line, from the line to the point.

How do we discuss area in a plane? Let us begin with this definition.

Modern Definition 2. The area of a rectangle is the product of the length of two adjacent sides. That is,

$$A = bh,$$

where b is the length of any chosen side, called the *base*, and h is the length of either adjacent side, called the *height*.

The area of an another plane region can be obtained by showing its relation to a rectangle. Recall these propositions from Euclid Book I.

Euclid Proposition I.34. *A diameter of a parallelogram bisects the area.*

Euclid Proposition I.36. *Parallelograms which are on equal bases and in the same parallels are equal to one another.*

Modern Proposition 2. *The area of a parallelogram is*

$$A = bh,$$

where b is the length of a side called the base*, and h is the distance from the opposite side to the base, called the* height*.*

Proof. Given a parallelogram with base of length b, their exists a rectangle on this base in the same parallels, whose area is bh. By Prop I.36, the parallelogram has the same area. $\square$

Now we move on to triangles.

Modern Definition 3. A *base* of a triangle is any one of its sides, or, is the length of a side also called the base.

The *height* of a triangle with given base is the distance from the vertex opposite the base to the line through the base.

Modern Proposition 3. *The area of a triangle is one-half of the base times the height; that is,*

$$A = \frac{1}{2}bh.$$

Proof. Consider a triangle $\triangle ABC$ with base $\overline{BC}$, so that $b = BC$. Let h be the distance from A to $\overleftrightarrow{BC}$.

Suppose that $m\angle CBA \geq m\angle BCA$. The line through C parallel to $\overleftrightarrow{AB}$, and the line through A parallel to $\overleftrightarrow{BC}$, intersect in a point; call this point D. Then $ABCD$ is a parallelogram of base b and height h, so its area is bh. By Euclid Prop I.34, the triangle is half of this area, so the area of the triangle if $\frac{1}{2}bh$. $\square$

3. Ratios from Book V

We directly interpret Euclid's ratios as modern fractions. That is, we take ratios to generally mean ratios of numbers, or magnitudes with the same units (so that the units cancel). Therefore $a : b :: c : d$ translates to $\frac{a}{b} = \frac{c}{d}$.

With this in mind, and using algebra which was not available to Euclid, we now rephrase and reinterpret some of the propositions from Book V which are used by Euclid in Book VI. These propositions are used in Euclid's proofs of Propositions VI.2, 3, 4.

Euclid Proposition V.9. *Magnitudes which have the same ratio to the same are equal to one another.*

Modern proof. This means that if $\frac{x}{a} = \frac{y}{a}$, then $x = y$. This is obtained via algebra by multiplying both sides by a. □

Euclid Proposition V.11. *Ratios which are the same with the same ratio are also the same with one another.*

Modern proof. This says that

$$\text{if} \quad \frac{a}{b} = \frac{c}{d} \quad \text{and} \quad \frac{c}{d} = \frac{e}{f}, \quad \text{then} \quad \frac{a}{b} = \frac{e}{f}.$$

When ratios are phrased as fractions, this is simply the transitive property of equality (Euclid Common Notion 1). □

Euclid Proposition V.16. *If four magnitudes be proportional, they will also be proportional alternately.*

Modern proof. This means

$$\frac{a}{b} = \frac{c}{d} \quad \Rightarrow \quad \frac{a}{c} = \frac{b}{d}.$$

This is obtained by multiplying both sides by $\frac{b}{c}$. □

Euclid Proposition V.22. *If there be six magnitudes which taken two and two together are in the same ratio, they will also be in the same ratio ex aequali.*

Modern proof. This means that

$$\frac{a}{b} = \frac{c}{d} \quad \text{and} \quad \frac{b}{c} = \frac{d}{e} \quad \Rightarrow \quad \frac{a}{c} = \frac{d}{f}.$$

To see this, multiply the first equation by the second to get

$$\frac{ab}{bc} = \frac{cd}{de}.$$

Now cancel the b's on the left and the d's on the right to get

$$\frac{a}{c} = \frac{c}{e}.$$

□

4. Book VI Propositions 1 - 4

Euclid Proposition VI.1. *Triangles which are under the same height are to one another as their bases.*

Modern proof. Let triangle 1 and triangle 2 be triangles with areas A_1, A_2 and bases b_1, b_2, respectively. We wish to show that

$$\frac{A_1}{A_2} = \frac{b_1}{b_2}.$$

Let h be the common height. Then

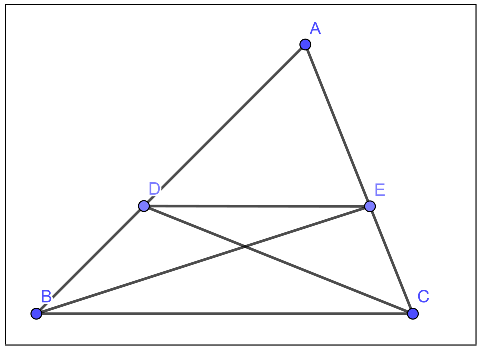
$$A_1 = \frac{1}{2}b_1 h \quad \text{and} \quad A_2 = \frac{1}{2}b_2 h.$$

Then $h = \dfrac{2A_1}{b_1}$ and $h = \dfrac{2A_2}{b_2}$. From this, $\dfrac{2A_1}{b_1} = \dfrac{2A_2}{b_2}$. Divide both sides by 2, divide both sides by A_2, and multiply both b_1 to arrive at $\dfrac{A_1}{A_2} = \dfrac{b_1}{b_2}$. □

Each of Euclid's Propositions VI.2 and VI.3 are presented as if and only if statements. We will prove each direction of implication separately. The converse of Euclid Proposition VI.4 is presented by Euclid in Proposition VI.5. We will use what are essentially Euclid's proofs, presented in two-columns and using some algebra to simplify the presentation where appropriate.

Euclid Proposition VI.2a. *If a straight line be drawn parallel to one of the sides of a triangle, it will cut the sides of the triangle proportionally.*

Proof. Let $\triangle ABC$ be given. Let D be a point on $\overline{AB}$ and E a point on $\overline{AC}$ such that $\overline{DE} \parallel \overline{BC}$.



We wish to show that

$$\frac{AD}{BD} = \frac{AE}{CE}.$$

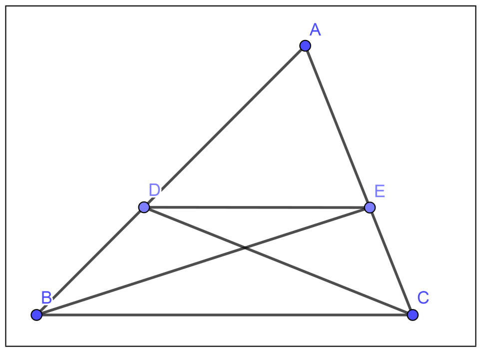
Join $\overline{DC}$ and $\overline{EB}$.

Claim	Reason
$\overline{DE} \parallel \overline{BC}$	Given
$\triangle DEB$ and $\triangle DEC$ share a common base	Let $\overline{DE}$ be the common base
$\triangle DEB$ and $\triangle DEC$ share a common height	Let $\overline{DE}$ to $\overline{BC}$ be the common height
area$(\triangle DEB)$ = area$(\triangle DEC)$	$\triangle DEB$ and $\triangle DEC$ have the same base and height
$\triangle ADE$ and $\triangle DEB$ share a common height	Let the line parallel to AB through E produce the height
$\frac{\text{area}(\triangle ADE)}{\text{area}(\triangle DEB)} = \frac{AE}{CE}$	Prop VI.1: $\triangle ADE$ and $\triangle DEB$ share a common height
$\triangle ADE$ and $\triangle DEC$ share a common height	Let the line parallel to AC through D produce the height
$\frac{\text{area}(\triangle ADE)}{\text{area}(\triangle DEC)} = \frac{AD}{BD}$	Prop VI.1: $\triangle ADE$ and $\triangle DEC$ share a common height
$\frac{\text{area}(\triangle ADE)}{\text{area}(\triangle DEB)} = \frac{AD}{BD}$	Substitution: area$(\triangle DEC)$ = area$(\triangle DEB)$)
$\frac{AD}{BD} = \frac{AE}{CE}$	Common Notion 1

$\square$

Euclid Proposition VI.2b. *If a straight line be drawn parallel through a triangle such that the sides of the triangle are cut proportionally, the line joining the points o section will be parallel to the remaining side of the triangle.*

Proof. Let $\triangle ABC$ be given. Let D be a point on $\overline{AB}$ and E a point on $\overline{AC}$ such that $\dfrac{AD}{AB} = \dfrac{AE}{CE}$.



We wish to show that

$$\overline{DE} \parallel \overline{BC}.$$

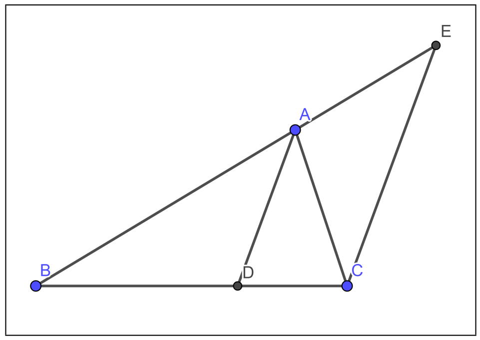
Join $\overline{DC}$ and $\overline{EB}$.

Claim	Reason
$\dfrac{AD}{AB} = \dfrac{AE}{CE}$	Given
$\triangle ADE$ and $\triangle DEB$ share a common height	Let the line parallel to AB through E produce the height
$\dfrac{\text{area}(\triangle ADE)}{\text{area}(\triangle DEB)} = \dfrac{AE}{CE}$	Prop VI.1: $\triangle ADE$ and $\triangle DEB$ share a common height
$\triangle ADE$ and $\triangle DEC$ share a common height	Let the line parallel to AC through D produce the height
$\dfrac{\text{area}(\triangle ADE)}{\text{area}(\triangle DEC)} = \dfrac{AD}{BD}$	Prop VI.1: $\triangle ADE$ and $\triangle DEC$ share a common height
$\dfrac{\text{area}(\triangle ADE)}{\text{area}(\triangle DEB)} = \dfrac{AD}{BD}$	Substitution: area($\triangle DEC$) = area($\triangle DEB$))
$\dfrac{\text{area}(\triangle ADE)}{\text{area}(\triangle DEB)} = \dfrac{\text{area}(\triangle ADE)}{\text{area}(\triangle DEC)}$	Common Notion 1
$\text{area}(\triangle DEB) = \text{area}(\triangle DEC)$	Algebra
$\triangle DEB$ and $\triangle DEC$ share a common base	Let $\overline{DE}$ be the common base
$\overline{DE} \parallel \overline{BC}$	Prop I.39: Same area and base imply same parallels

□

Euclid Proposition VI.3a. *If an angle of a triangle be bisected and the straight line cutting the angle cut the base also, the segments of the base will have the same ratio as the remaining sides of the triangle.*

Proof. Let $\triangle ABC$ be given. Let D be a point on $\overline{BC}$ such that $\overline{AD}$ bisects $\angle BAC$.



We wish to show that

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

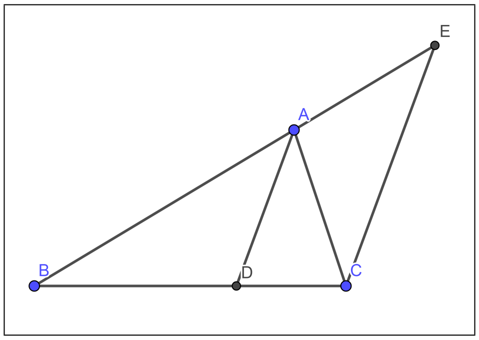
Let E be the point of intersection of $\overleftrightarrow{AB}$ and the line through C parallel to $\overline{AD}$. Join $\overline{AE}$ and $\overline{CE}$.

Claim	Reason
$\overline{AD} \parallel \overline{CE}$	Construction
$\frac{BD}{CD} = \frac{AB}{AE}$	Prop VI.2
$\overline{AD}$ bisects $\angle BAC$	Given
$\angle BAD \cong \angle CAD$	Definition of bisect
$\angle ACE \cong \angle CAD$	Prop I.29: Alternate Interior Angles
$\angle ACE \cong \angle BAD$	Common Notion 1
$\angle BAD \cong \angle BEC$	Prop I.29: Consecutive Angles
$\angle ACE \cong \angle BEC$	Common Notion 1
$AC = AE$	Prop I.6: Converse of Base Angle Theorem
$\frac{BD}{CD} = \frac{AB}{AC}$	Substitution into Row 2

□

Euclid Proposition VI.3b. *If an angle of a triangle is cut by a straight line which cuts the base also, and the segments of the base have the same ratio as the remaining sides of the triangle, the straight line joined from the vertex to the point of section will bisect the angle of the triangle.*

Proof. Let $\triangle ABC$ be given. Let D be a point on $\overline{BC}$ such that $\frac{BD}{CD} = \frac{AB}{AC}$. Join AD.



We wish to show that

$$\angle BAD \cong \angle CAD.$$

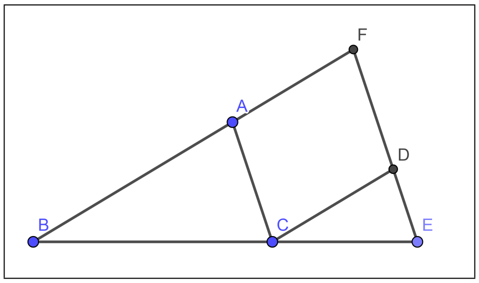
Let E be the point of intersection of $\overleftrightarrow{AB}$ and the line through C parallel to $\overline{AD}$. Join $\overline{AE}$ and $\overline{CE}$.

Claim	Reason
$\overline{AD} \parallel \overline{CE}$	Construction
$\frac{BD}{CD} = \frac{AB}{AE}$	Prop VI.2
$\frac{BD}{CD} = \frac{AB}{AC}$	Given
$AC = AE$	Common Notion 1 and Prop V.9
$\angle ACE \cong \angle BEC$	Prop I.5: Base Angle Theorem
$\angle ACE \cong \angle CAD$	Prop I.29: Alternate Interior Angles
$\angle BEC \cong \angle CAD$	Common Notion 1
$\angle BAD \cong \angle BEC$	Prop I.29: Consecutive Angles
$\angle BAD \cong \angle CAD$	Common Notion 1
$\overline{AD}$ bisects $\angle BAC$	Definition of bisect

$\square$

Euclid Proposition VI.4. *In similar triangles the sides about the equal angles are proportional.*

Proof. Given similar triangles, position them so that they share a point, as follows. Let $\triangle ABC$ and $\triangle DCE$ be given, with $\triangle ABC \sim \triangle DCE$. Join $\overline{AD}$.



We wish to show that

$$\frac{AB}{BC} = \frac{CD}{CE}.$$

Let F be the point of intersection of $\overleftrightarrow{AB}$ and the line through C parallel to $\overline{AD}$. Join $\overline{AF}$ and $\overline{CF}$.

Claim	Reason
$\angle ABC \cong \angle DCE$	Given
$\overline{DC} \parallel \overline{BF}$	Prop I.28: Consecutive Angles
$\angle ACB \cong \angle DEC$	Given
$\overline{AC} \parallel \overline{EF}$	Prop I.28: Consecutive Angles
Quadrilateral $ACDF$ is a parallelogram	Definition of parallelogram
$AF = CD$	Prop I.34 + Row 5
$\frac{AB}{AF} = \frac{BC}{CE}$	Prop VI.2 + Row 4
$\frac{AB}{CD} = \frac{BC}{CE}$	Substitution
$\frac{AB}{BC} = \frac{CD}{CE}$	Prop V.16

Analogously, the other pairs of sides about equal angles are proportional. $\square$

Modern Proposition 4. *Triangle Similarity Theorem If $\triangle ABC \sim \triangle DEF$, then*

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}.$$

Proof. From Euclid Proposition VI.4, we have

$$\frac{AB}{AC} = \frac{DE}{DF}, \quad \frac{AB}{BC} = \frac{DE}{EF}, \quad \text{and} \quad \frac{AC}{BC} = \frac{DF}{EF}.$$

Apply Euclid V.16 to the first equation to get $\dfrac{AB}{DE} = \dfrac{AC}{DF}$.

Apply Euclid V.16 to the second equation to get $\dfrac{AB}{DE} = \dfrac{BC}{EF}$.

Apply Common Notion 1 to get $\dfrac{AC}{DF} = \dfrac{BC}{EF}$. $\square$

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